Hernández Lecture 1

Tuesday, June 7, 2022 10:14 AM

hernandez@ku.edu ← email notes

Algebraic Hypersurfaces

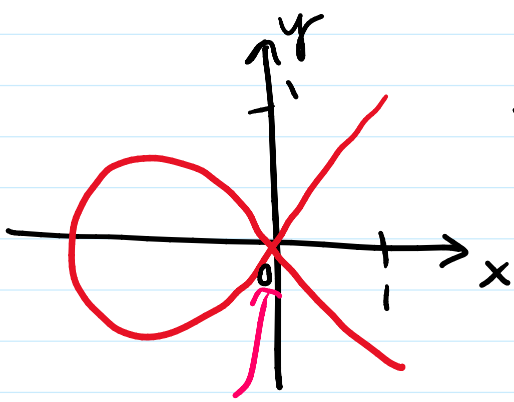
- k is a field ($\mathbb{Q}, \mathbb{R}, \mathbb{C}$, or $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ usually)
- $f = f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$
- $\mathbb{V}(f) = \{\vec{a} \in k^n : f(\vec{a}) = f(a_1, \dots, a_n) = 0\}$

↑ vanishing set

We call this the (affine algebraic) hypersurface defined by f.

Q. How complicated can $\mathbb{V}(f)$ be in a nbhd of a singular pt?

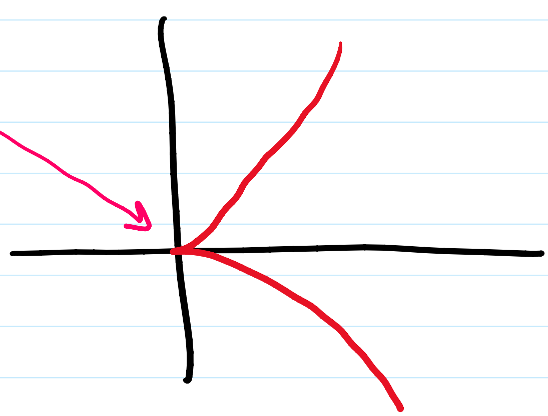
↑ Sharp bends cusps

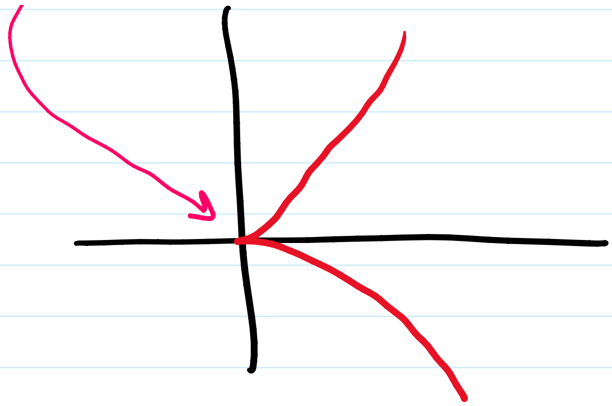


$y^2 - x^3 - x^2 = 0$

$k = \mathbb{R}$

singular pt





(Hauser's Gallery of Singular Algebraic Surfaces)

It's hard to graphically recognize cusps in $\mathbb{F}_p$ & singularities

Slides & problems on his website.

$f \in K[x] = K[x_1, \dots, x_n]$

$\cancel{\vec{a}}\ \underline{0} \in \mathbb{V}(f) \leftrightarrow f(\cancel{\vec{a}}\ \underline{0}) = 0$

When is f singular at $\cancel{\vec{a}}\ \underline{0}$?

(can always translate to origin)

$f(\underline{0}) = 0 \leftrightarrow f \in \mathfrak{m} = \langle x_1, \dots, x_n \rangle$ (no constant term)

max ideal

Definition $T_0 V$ = tangent space to V at $\underline{0}$

$(V = \mathbb{V}(f))$ = {all lines thru $\underline{0}$ & are tangent to V}

Let ℓ be a line thru the origin. When is it tangent to V?

- $\ell = \{at : t \in K\}$ $\quad \underline{0} \neq a = (a_1, \dots, a_n) \in K^n$
- $f = \cancel{f(\underline{0})}\ 0 + L + G$

• $f = \cancel{f(\underline{0})} + L + G$

$\uparrow$ linear term $\quad \uparrow$ grande (things of at least degree 2) higher-degree component, i.e. $G \in \mathfrak{m}^2$

• $f(at) = \underbrace{L(at)} + \underbrace{G(at)}$

point on line

$= t\,L(a) + t^2 \mid G(at)$

it's linear!

divides

<u>Want</u>: $t^2 \mid f(at)$ Why? What it means to divide tangentially twice (let secant lines get smaller & smaller)

so ℓ is tangent to V at $\underline{0}$ iff $L(\underline{a}) = 0$

$T_0 V = \{$ lines thru $\underline{0}$ tangent to V at $\underline{0} \}$

$= \{ a \in k^n : L(\underline{a}) = 0 \}$ (pt determines the line through $\underline{0}$)

$\rightsquigarrow$ from Linear Algebra we get:

$$\dim T_0 V = \begin{cases} n & \text{when } L = 0 \\ n-1 & \text{when } L \neq 0 \end{cases}$$

$\leftarrow$ so this is when we have singularity (too many tangent lines)

So f is Singular at $\underline{0}$ ($\underline{0}$ is a sing. pt of f) is $L = 0 \in k[x_1, \ldots, x_n]$

$L = 0 \in k[x_1, \dots, x_n]$

Notice we haven't used alg. closed for k.

$\boxed{\text{Check}} \Rightarrow L = \frac{\partial f}{\partial x_1}(\underline{0}) \cdot x_1 + \cdots + \frac{\partial f}{\partial x_n}(\underline{0}) \cdot x_n$

f is singular at $\underline{0}$ $\Leftrightarrow f(\underline{0}) = \frac{\partial f}{\partial x_1}(\underline{0}) = \cdots = \frac{\partial f}{\partial x_n}(\underline{0})$ $(=0)$

$\Leftrightarrow f \in \mathfrak{m}^2$

variety

LCB 121 - Daniel

LCB 323 - Marissa

(Can split time)