Hernández Lecture 2

Thursday, June 9, 2022 8:59 AM

Recap: · $f \in K[x_1, \dots, x_n]$, $f \neq 0$

K ← field ($\mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$)

· $V = \mathbb{V}(f) = \{a \in K^n : f(a) = 0\}$

V ← hypersurface

Assume $\underline{0} \in K^n$ in in V (i.e. $f(\underline{0}) = 0$)

$\underline{0}$ ← origin

Q: When is V/f singular at 0? ← answered last time

(How singular?) ← today

$\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subseteq K[x_1, \dots, x_n]$

$\mathfrak{m}$ ← "maximal ideal corres. to $\underline{0}$"

$$\dots \bigcap_{i=0}^{\infty} \mathfrak{m}^i \subseteq \cdots \subseteq \mathfrak{m}^t \subseteq \cdots \subseteq \mathfrak{m}^3 \subseteq \mathfrak{m}^2 \subseteq \mathfrak{m}$$

$\bigcap_{i=0}^{\infty} \mathfrak{m}^i = 0$ ← by exercises

$f \in \mathfrak{m}^2$ ⇐ (if singular) $f \in \mathfrak{m}$

$\Rightarrow \exists t$ s.t. $f \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$

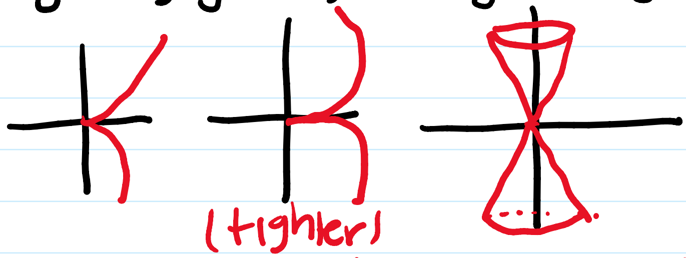
$\mathrm{mult}_{\underline{0}}(f) = \mathrm{mult}(f) = \max\{t : f \in \mathfrak{m}^t\}$

If $f \in \mathfrak{m}^t \Rightarrow$ lowest monomials are of at least degree t

$\mathrm{mult}(f) = t \Rightarrow$ lowest monomial of f is degree t

$\underline{\text{Ex}}$ If $f \in \{y^2 - x^3, y^2 - x^5, x^2 + y^2 - z^2\} \Rightarrow \mathrm{mult}(f) = 2$

Ex If $f \in \{y^2 - x^3,\ y^2 - x^5,\ x^2 + y^2 - z^2\} \Rightarrow mult(f) = 2$



(tighter)

mult doesn't distinguish between these 2 singularities. So let's explore better ways

$\frac{1}{mult(f)}$ satisfies
- in $(0,1] \cap \mathbb{Q}$
- $= 1$ when f is nonsingular $(\Leftrightarrow)$
- "worse singularities" $\leftrightarrow$ "smaller values"

this associates a # to shape but we could also assoc. an ideal & look at how "complicated" it is

Idea: Refine this

(calculus side) $\mathbb{R}$, $\mathbb{C}$	$\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$
· differentiation · integration ⋮ · resolution of singularities	· Frobenius map

$\left.\begin{array}{l} \text{differentiation} \\ \text{integration} \\ \vdots \\ \text{resolution of singularities} \end{array}\right\}$ tools we can use

no longer have Calc I tools in this case.

Goal: Introduce Frobenius & use it to distinguish singularities

take $p \rightarrow \infty$ & try to get to $\mathbb{R}/\mathbb{C}$ side

---

· $f \in \mathbb{F}[x_1, \dots, x_n] = R$

$\cdot\, f \in \mathbb{F}_p[x_1, \dots, x_n] = R$

$0 \le i \le p$

$\binom{p}{i} = \frac{p!}{i!(p-i)!} \equiv 0 \mod p$ if $0 < i < p$ (p on numerator won't get canceled)

$\Rightarrow g, h \in R \Rightarrow (g+h)^p = \sum_{i=0}^{p} \binom{p}{i} g^i h^{p-i}$ (Binomial Thm)

great consequence of this fact $= g^p + h^p$

$F: R \to R$ defined by $F(g) = g^p$ Frobenius map/endomorph (morph from space to itself)

p^{th} power map

$\left.\begin{array}{l} \cdot\, F(g+h) = F(g) + F(h) \\ \cdot\, F(gh) = F(g)F(h) \end{array}\right\} \Rightarrow$ Ring hom.

$R \xrightarrow{F} F(R) \subseteq R$, $F(R) = \{g^p : g \in R\}$ subring

injective b/c in a domain (can't have $g^p = 0$)

& forced to be onto

$\Rightarrow \cong$

The Frobenius map is an iso. onto its image!

So we have an iso. from R to a subring of R that's iso to R (heading to fractals)

$g \in R \Rightarrow g = \sum_{u \in \mathbb{N}^n} \alpha_u x^u$ (finite sum) α_u in $\mathbb{F}_p$

$u = (u_1, \dots, u_n)$

$x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$

$\Rightarrow g^p = \left(\sum \alpha_u x^u\right)^p = \sum \alpha_u^p x^p = \sum \alpha_u x^p = \sum \alpha_u (x_1^{pu_1} \cdots x_n^{pu_n}) \in \mathbb{F}_p[x_1^p, \dots, x_n^p]$

Fermat's

Fermat's Little Thm $\qquad\qquad \in \mathbb{F}_p[x_1^p, \dots, x_n^p]$

Thus our $R \xrightarrow{F} F(R)$ is $\mathbb{F}_p[x_1,\dots,x_n] \xrightarrow{F} \mathbb{F}_p[x_1^p,\dots,x_n^p] \subseteq \mathbb{F}_p[x_1,\dots,x_n]$

<u>Examine $\mathbb{F}_p[x_1^p,\dots,x_n^p] \subseteq \mathbb{F}_p[x_1,\dots,x_n]$:</u>

$\boxed{n=1 \text{ case}}$ $\underbrace{\mathbb{F}_p[x^p]}_{\text{"scalars"}} \subseteq \underbrace{\mathbb{F}_p[x]}_{\text{"vectors"}}$ (analogous how $\mathbb{R} \subseteq \mathbb{C}$ where $\mathbb{C}$ is a v.s. over $\mathbb{R}$)

$$g \in \mathbb{F}_p[x] \Rightarrow g = \sum_{i=0}^{d} \alpha_i x^i$$

Apply Division Algorithm: $i = s_i p + r_i$, $s_i, r_i \in \mathbb{N}$, $0 \le r_i < p$

$$= \sum_{i=0}^{d} \alpha_i (x^p)^{s_i} x^{r_i}$$

Gather all possible remainders $r_1, \dots, r_N$

$$g = \underbrace{\boxed{\sum \alpha (x^p)^{s_i}}}_{\substack{\text{remainder} \\ = r_1}} x^{r_1} + \cdots + \underbrace{\boxed{\sum \alpha (x^p)^{s_i}}}_{\substack{\text{remainder} \\ = r_N}} x^{r_N}$$

in $\mathbb{F}_p[x^p]$

$\Rightarrow g$ is an $\mathbb{F}_p[x^p]$-linear combo of $\underbrace{1, x, \dots, x^{p-1}}_{\Rightarrow \text{ basis for } \mathbb{F}_p[x] \text{ over } \mathbb{F}_p[x^p]}$

$\boxed{n>1 \text{ case}}$ $\mathbb{F}_p[x_1^p, \dots, x_n^p] \subseteq \mathbb{F}_p[x_1, \dots, x_n]$ has "basis" consisting of monomials of the form $x_1^{a_1} \cdots x_n^{a_n}$ w/ all $0 \le a_i < p$

$\mathbb{F}_p[\underline{x}^p] \subseteq \mathbb{F}_p[\underline{x}]$

$\mathbb{F}_p[\underline{x}^p] \subseteq \mathbb{F}_p[\underline{x}]$

very nice ext. of rings

also has its own Frobenius

$\Rightarrow \cdots \subseteq \mathbb{F}_p[\underline{x}^{p^2}] \subseteq \mathbb{F}_p[\underline{x}^p] \subseteq \mathbb{F}_p[\underline{x}]$ Frobenius fractal

each containment are all iso & zooming in & see an iso. obj.

$$\cdots \subseteq R^{p^3} \subseteq R^{p^2} \subseteq R^p \subseteq R$$

$$\cdots \subseteq S^{p^2} \subseteq S^p \subseteq S$$

self-similar chain of rings

$\mathbb{Q}[x^2] \subseteq \mathbb{Q}[x]$

$f^2 \leftarrow f$

not a ring map!

not gonna be self-similar (this is why we look at $\mathbb{F}_p$)

Can we go to the right? Yes. $\mathbb{F}_p[\underline{x}] \subseteq \mathbb{F}_p[\underline{x}^{1/p}]$

(worksheet)

$\mathbb{F}_p[x] \subseteq \mathbb{F}_p[x] \subseteq \overline{\mathbb{F}_p[x]}$

$t^p - x \in \overline{\mathbb{F}_p[x]}[t]$

need to show $x^{1/p}$ is!